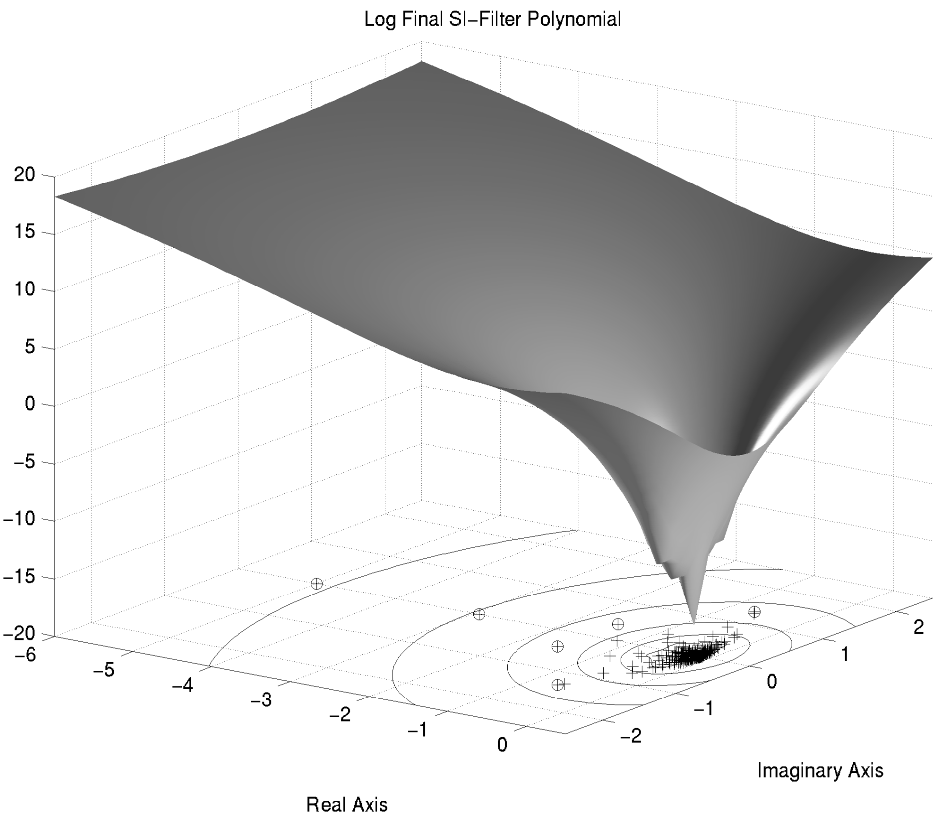

Figure 4.9: Total Filter Polynomial with Spectral Transformation

If $\mathbf{M}$ is singular then the operator $\mathbf{S} \equiv (\mathbf{A} - \sigma\mathbf{M})^{-1}\mathbf{M}$ has a non-trivial null space and the bilinear function $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^H\mathbf{M}\mathbf{y}$ is a semi-inner product and $\|\mathbf{x}\|_M \equiv \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ is a semi-norm. Since $\mathbf{A} - \sigma\mathbf{M}$ is assumed to be nonsingular, $\mathcal{N} \equiv Null(\mathbf{S}) = Null(\mathbf{M})$. Vectors in $\mathcal{N}$ are generalized eigenvectors corresponding to *infinite* eigenvalues. Typically, one is only interested in the finite eigenvalues of $(\mathbf{A}, \mathbf{M})$ and these will correspond to the non-zero eigenvalues of S. The invariant subspace corresponding to these non-zero eigenvalues is easily corrupted by components of vectors from $\mathcal{N}$ during the Arnoldi process. However, using the M-Arnoldi process with some refinements can provide a solution.

In order to better understand the situation, it is convenient to note that since $\mathbf{M}$ is positive semi-definite, there is an orthogonal matrix $\mathbf{Q}$ such that

$$\mathbf{Q}^H\mathbf{M}\mathbf{Q} = \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

where $\mathbf{D}$ is a positive definite diagonal matrix of order n, say. Thus

$$\hat{\mathbf{S}} \equiv \mathbf{Q}^H\mathbf{S}\mathbf{Q} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{0} \\ \mathbf{S}_2 & \mathbf{0} \end{bmatrix},$$

where $\mathbf{S}_1$ is a square matrix of order n and $\mathbf{S}_2$ is an $m \times n$ matrix with the original $\mathbf{A}, \mathbf{M}$ being of order $m + n$. Observe now that a non-zero eigenvalue ν of $\hat{\mathbf{S}}$ satisfies $\hat{\mathbf{S}}\mathbf{x} = \mathbf{x}\nu$, i.e.

$$\begin{bmatrix} \mathbf{S}_1\mathbf{x}_1 \\ \mathbf{S}_2\mathbf{x}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1\theta \\ \mathbf{x}_2\theta \end{bmatrix}$$

so that $\mathbf{x}_2 = \theta^{-1}\mathbf{S}_2\mathbf{x}_1$ must hold. Note also that for any eigenvector $\mathbf{x}^H = (\mathbf{x}_1^H, \mathbf{x}_2^H)$, the leading vector $\mathbf{x}_1$ must be an eigenvector of $\mathbf{S}_1$. Since $\hat{\mathbf{S}}$ is block triangular, $\sigma(\hat{\mathbf{S}}) = \sigma(\mathbf{S}_1) \cup \sigma(\mathbf{0}_m)$. Assuming $\mathbf{S}_2$ has full rank, it follows that if $\mathbf{S}_1$ has a zero eigenvalue then there is no corresponding eigenvector (since $\mathbf{S}_2\mathbf{x}_1 = \mathbf{0}$ would be implied). Thus, if zero is an eigenvalue of $\mathbf{S}_1$ with algebraic multiplicity m_o, then zero is an eigenvalue of $\hat{\mathbf{S}}$ of algebraic multiplicity $m + m_o$ and with geometric multiplicity m. Of course, since $\mathbf{S}$ is similar to $\hat{\mathbf{S}}$ all of these statements hold for $\mathbf{S}$ as well.

4.5.2 Eigenvector/Null-Space Purification

With these observations in hand, it is possible to see the virtue of using M-Arnoldi on $\mathbf{S}$. After k-steps of M-Arnoldi,

$$\mathbf{S}\mathbf{V} = \mathbf{V}\mathbf{H} + \mathbf{f}\mathbf{e}_k^T \quad \text{with} \quad \mathbf{V}^H\mathbf{M}\mathbf{V} = \mathbf{I}_k, \mathbf{V}^H\mathbf{M}\mathbf{f} = \mathbf{0}.$$

Introducing the similarity transformation $\mathbf{Q}$ gives

$$\hat{\mathbf{S}}\hat{\mathbf{V}} = \hat{\mathbf{V}}\mathbf{H} + \hat{\mathbf{f}}\mathbf{e}_k^T \quad \text{with} \quad (\mathbf{Q}\hat{\mathbf{V}})^H\mathbf{M}\mathbf{Q}\hat{\mathbf{V}} = \mathbf{I}_k, (\mathbf{Q}\hat{\mathbf{V}})^H\mathbf{M}\mathbf{Q}\hat{\mathbf{f}} = \mathbf{0},$$